The Generalized Mayer-Vietoris Principle

We would introduce "Cech cohomology" and show how it relates to the de Rahm cohomology.

We would suppose there exists a cover $\{U_i\}_{i\in I}$ of M with countable index I. and get sequence below

$$M \leftarrow \coprod U_{\alpha_0} \;\underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}}\; \coprod_{\alpha_0<\alpha_1} U_{\alpha_0\alpha_1} \;\underset{\partial_2}{\overset{\partial_0,\ \partial_1}{\Lleftarrow}}\; \coprod_{\alpha_0<\alpha_1<\alpha_2} U_{\alpha_0\alpha_1\alpha_2} \;\leftleftarrows\!\!\leftleftarrows \cdots$$

where $U_{\alpha_0\alpha_1} = U_{\alpha_0}\cap U_{\alpha_1}$, generally, $U_{\alpha_0\cdots\alpha_j} = \bigcap U_{\alpha_i}$

∂_i is the inclusion that "ignores" the i-th index.

$\partial_i : U_{\alpha_0\cdots\alpha_i\cdots\alpha_j} \hookrightarrow U_{\alpha_0\cdots\hat{\alpha}_i\cdots\alpha_j}$.

From the inclusion of space, we have a chain of forms.

tions of forms

$$\Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \;\underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}}\; \prod_{\alpha_0<\alpha_1} \Omega^*(U_{\alpha_0\alpha_1}) \;\underset{\delta_2}{\overset{\delta_0,\ \delta_1}{\Rrightarrow}}\; \prod_{\alpha_0<\alpha_1<\alpha_2} \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \;\rightrightarrows\!\!\rightrightarrows \cdots$$

Since we used to define only $U_{\alpha\beta\gamma}$ $(\alpha<\beta<\gamma)$. To generalize our notation, we use the convention: $\omega_{\dots\alpha\dots\beta\dots} = -\omega_{\dots\beta\dots\alpha\dots}$

In particular, a form with repeat indeces is 0.

Now we define $\delta = \sum (-1)^i \delta_i$, for example

$\delta : \prod \Omega^*(U_{\alpha_0\alpha_1}) \to \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2})$, exactly, we have

$(\delta\xi)_{\alpha_0\alpha_1\alpha_2} = \xi_{\alpha_1\alpha_2} - \xi_{\alpha_0\alpha_2} + \xi_{\alpha_0\alpha_1}$

$\Downarrow$

component of $\delta\xi$ in $\Omega^*(U_{\alpha_0\alpha_1\alpha_2})$.

Defination: If $\omega \in \prod \Omega^q(U_{\alpha_0\cdots\alpha_p})$, ω has component $\omega_{\alpha_0\cdots\alpha_p} \in \Omega^q(U_{\alpha_0\cdots\alpha_p})$ and

$$(\delta\omega)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_{p+1}}$$

Proposition: $\delta^2 \equiv 0$. We skip the proof since it's simple.

With notations and properties above, we can state the first non-so-trivial proposition.

Proposition 8.5. (The Generalized Mayer–Vietoris Sequence). *The sequence*

$$0 \longrightarrow \Omega^*(M) \xrightarrow{r} \prod \Omega^*(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega^*(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots$$

is exact; in other words, the δ-cohomology of this complex vanishes identically.

Proof: $r(\Omega^*(M)) = \ker\left(\prod \Omega^*(U_\alpha) \to \prod \Omega^*(U_{\alpha_0\alpha_1})\right)$

is obvious

Let $\{\rho_\alpha\}$ be a partition of unity related to $\{U_\alpha\}$.

$\omega \in \prod \Omega^*(U_{\alpha_0 \cdots \alpha_p})$ a p-cocycle.

Define: $\tau_{\alpha_0 \cdots \alpha_{p-1}} = \sum \rho_\alpha \omega_{\alpha\alpha_0 \cdots \alpha_{p-1}}$

Then $(\delta\tau)_{\alpha_0 \cdots \alpha_p} = \sum_i (-1)^i \tau_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p} = \sum (-1)^i \rho_\alpha \omega_{\alpha\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p}$

Since ω is a cocycle.

$$(\delta\omega)_{\alpha\alpha_0 \cdots \alpha_p} = \omega_{\alpha_0 \cdots \alpha_p} + \sum (-1)^{i+1} \omega_{\alpha\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p} = 0.$$

hence $(\delta\tau)_{\alpha_0 \cdots \alpha_p} = \sum (-1)^i \rho_\alpha \omega_{\alpha\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_p} = \sum \rho_\alpha \omega_{\alpha_0 \cdots \alpha_i \cdots \alpha_p}$

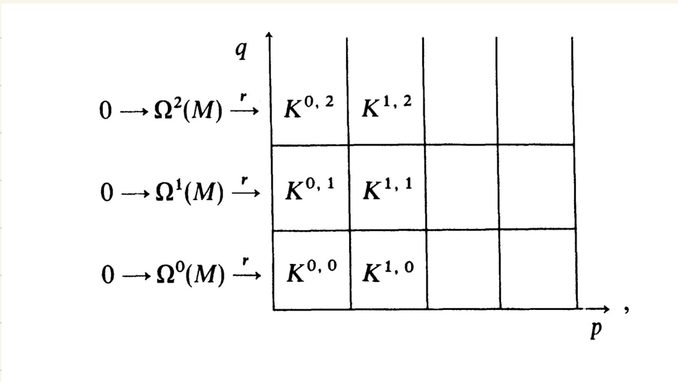
$= \omega_{\alpha_0 \cdots \alpha_i \cdots \alpha_p}$

Hence every cocycle is a coboundary. The sequence is exact. $\square$

A double complex is a diagram of shape $\mathbb{Z}_{\geq} \times \mathbb{Z}_{\geq}$ such that each row and column is a complex and all the squares commute.

We have a double complex below.

Note $K^{p,q} = C^p(\mathcal{U}, \Omega^q) = \prod \Omega^q(U_{\alpha_0 \cdots \alpha_p})$



We call $C^*(\mathcal{U}, \Omega^*) = \bigoplus_{p,q \geq 0} C^p(\mathcal{U}, \Omega^q)$, the Ceck-de Rahm complex with chain map

$$D : \bigoplus_{p+q=n} C^p(\mathcal{U}, \Omega^q) \to \bigoplus_{p+q=n+1} C^p(\mathcal{U}, \Omega^q)$$

such that $D|_{K^{p,q}} = \delta + (-1)^p d$

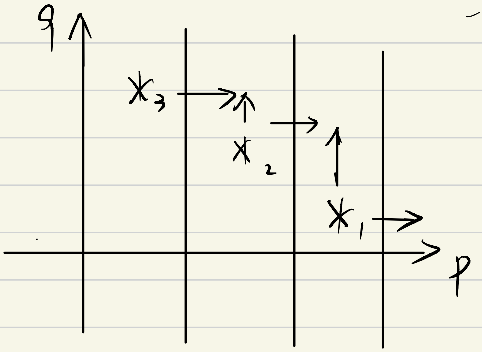
Now we will prove the Generalized M-V principle:

The restriction $r: \Omega^*(M) \to C^*(\mathcal{U}, \Omega^*)$ induce an isomorphism

$$r^*: H^*_{DR}(M) \to H^*_D(C^*(\mathcal{U}, \Omega^*))$$ in cohomology

Proof: $Dr = (\delta + d) r = dr = rd$, thus r is a chain map.

Assume ϕ is a cocycle relative to D. By δ-exactness the lowest



component of ϕ is δ of something. By substracting D(something) from ϕ, we can remove the lowest component of ϕ and still stay in the same cohomology class as ϕ. After iterating this procedure enough times we can remove in its cohomology class to a cocycle ϕ' with only the top component, ϕ' is a closed global form because $d\phi' = 0$ and $\delta\phi' = 0$.

Injective: If $r(\omega) = D\phi$, we can shorten ϕ as before by substracting boundaries until it consists of only top component. Then because $\delta\phi$ is 0, it is a global form on M. So ω is exact.

$\square$

Notice the skill we use in the proof of genelized M-V principle.

The proof of this proposition is a very general argument from which we may conclude: *if all the rows of an augmented double complex are exact, then the D-cohomology of the complex is isomorphic to the cohomology of the initial column.*

ınctions on the $(p+1)$-fold intersections $U_{\alpha_0 \ldots \alpha_p}$.

$$
\begin{array}{l|l|l|l}
0 \to \Omega^2(M) \xrightarrow{r} & \prod \Omega^2(U_{\alpha_0}) & & \\
0 \to \Omega^1(M) \to & \prod \Omega^1(U_{\alpha_0}) & & \\
0 \to \Omega^0(M) \to & \prod \Omega^0(U_{\alpha_0}) & \prod \Omega^0(U_{\alpha_0 \alpha_1}) & \prod \Omega^0(U_{\alpha_0 \alpha_1 \alpha_2}) \\
\hline
 & i\uparrow & i\uparrow & i\uparrow \qquad p \\
 & C^0(\mathfrak{U}, \mathbb{R}) \to & C^1(\mathfrak{U}, \mathbb{R}) \to & C^2(\mathfrak{U}, \mathbb{R}) \to \\
 & \uparrow & \uparrow & \uparrow \\
 & 0 & 0 & 0
\end{array}
$$

The bottom row

The sequence of Kernel of bottom δ, denoted as $C^*(\mathfrak{U}, \mathbb{R})$ is is a differential complex, and the homology of the complex, $H^*(\mathfrak{U}, \mathbb{R})$ is the Cech cohomology of the cover $\mathfrak{U}$.

If the augmented column is exact, we directly get

$$H^*(\mathfrak{U}, \mathbb{R}) \cong C^*(\mathfrak{U}, \Omega^*) \cong H^*_{DR}(M)$$

The failure of p^{th} column to be exact is measured by the homology group $\prod_{q\geq 1} H^q(U_{\alpha_0\cdots\alpha_p})$. Hence a good cover gives

$$H^*(\mathcal{U}, \mathbb{R}) \cong H^*_{DR}(M)$$

Corallary 1. All good covers induce the same cohomology.

Proof: All Cech-de Rahm cohomology of good cover is isomorphism to de-Rahm cohomology.

Corallary 2. If M holds a finite good cover, the Cech-de Rahm cohomology is finite-dimension.

Corollary 3: If M is compact, then the Cech-de Rahm cohomology is finite dimension.

Reference

Bott, Tu. Differential forms in Algebraic Topology